Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2946. [2004 : 230, 233] *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*

Let x, y, z be positive real numbers satisfying $x^2+y^2+z^2=1$. Prove that

(a) $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-(x+y+z)\geq 2\sqrt{3}$.

(b) $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+(x+y+z)\geq 4\sqrt{3}$.

I. *Solution by Arkady Alt, San Jose, CA, USA.*

(a) Note first that, for any $u>0$, the inequality $\frac{1}{u}-u\geq\frac{4\sqrt{3}}{3}-2\sqrt{3}u^2$ is equivalent to each of the following:

$$\begin{aligned} 6\sqrt{3}u^3-3u^2-4\sqrt{3}u+3 &\geq 0\,, \\ 2(\sqrt{3}u)^3-(\sqrt{3}u)^2-4(\sqrt{3}u)+3 &\geq 0\,, \\ (\sqrt{3}u-1)^2(2\sqrt{3}u+3) &\geq 0\,. \end{aligned}$$

The last inequality is clearly true. Hence,

$$\begin{aligned} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-(x+y+z) &= \sum_{\text{cyclic}}\left(\frac{1}{x}-x\right) \\ &\geq \sum_{\text{cyclic}}\left(\frac{4\sqrt{3}}{3}-2\sqrt{3}x^2\right) = 2\sqrt{3}\,. \end{aligned}$$

(b) For any $u>0$, the inequality $\frac{1}{u}+u\geq\frac{5\sqrt{3}}{3}-\sqrt{3}u^2$ is equivalent to each of the following:

$$\begin{aligned} 3\sqrt{3}u^3+3u^2-5\sqrt{3}u+3 &\geq 0\,, \\ (\sqrt{3}u)^3+(\sqrt{3}u)^2-5(\sqrt{3}u)+3 &\geq 0\,, \\ (\sqrt{3}u-1)^2(\sqrt{3}u+3) &\geq 0\,. \end{aligned}$$

The last inequality is clearly true. Hence,

$$\begin{aligned} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+(x+y+z) &= \sum_{\text{cyclic}}\left(\frac{1}{x}+x\right) \\ &\geq \sum_{\text{cyclic}}\left(\frac{5\sqrt{3}}{3}-\sqrt{3}x^2\right) = 4\sqrt{3}\,. \end{aligned}$$

II. *Solution by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

(a) By the Root–Mean–Square Inequality, we have

$$\frac{x+y+z}{3} \;\le\; \sqrt{\frac{x^2+y^2+z^2}{3}} \;=\; \frac{1}{\sqrt{3}}$$

and hence,

$$x+y+z \;\le\; \sqrt{3}\,. \tag{1}$$

By the AM–HM Inequality, we have

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \;\ge\; \frac{9}{x+y+z} \;\ge\; 3\sqrt{3}\,. \tag{2}$$

From (1) and (2), the claim follows.

(b) We apply the AM–GM Inequality twice:

$$\begin{aligned}
&\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+(x+y+z)\\
&\qquad \ge\; 4\left(\frac{x+y+z}{xyz}\right)^{1/4} \;=\; 4\left(\frac{(x+y+z)(x^2+y^2+z^2)}{xyz}\right)^{1/4}\\
&\qquad =\; 4\left(\frac{x^2}{yz}+\frac{y}{z}+\frac{z}{y}+\frac{y^2}{zx}+\frac{z}{x}+\frac{x}{z}+\frac{z^2}{xy}+\frac{x}{y}+\frac{y}{x}\right)^{1/4}\\
&\qquad \ge\; 4\left(9\sqrt[9]{\frac{x^4y^4z^4}{x^4y^4z^4}}\right)^{1/4} \;=\; 4\sqrt[4]{9} \;=\; 4\sqrt{3}\,.
\end{aligned}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Bristol, UK; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; VEDULA N. MURTY, Dover, PA, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; IAN VANDERBURGH, University of Waterloo, Waterloo, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; ROGER ZARNOWSKI, Angela State University, San Angela, TX, U.S.A., YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer. About two thirds of the submitted solutions used one or more of the following: AM–GM Inequality, AM–HM Inequality, AM–RMS Inequality, Cauchy–Schwarz Inequality. The rest used calculus, convexity and Jensen's Inequality. One solver used the method of Lagrange's Multipliers.

Many solvers remarked that equality holds in either of the two inequalities if and only if $x=y=z=1/\sqrt{3}$.

Bencze obtained the generalization that if $x_k > 0$ *(for* $k = 1, 2, \ldots, n$*) such that* $\sum\limits_{k=1}^{n} x_k^2 = 1$, *then for all* $a, b > 0$,

$$a\left(\sum_{k=1}^{n}\frac{1}{x_k}\right) \pm b\left(\sum_{k=1}^{n} x_k\right) \;\ge\; (an \pm b)\sqrt{n}\,.$$